# Prime Number Theorem

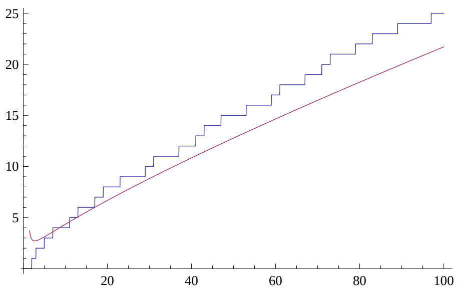
The Prime Number Theorem states that

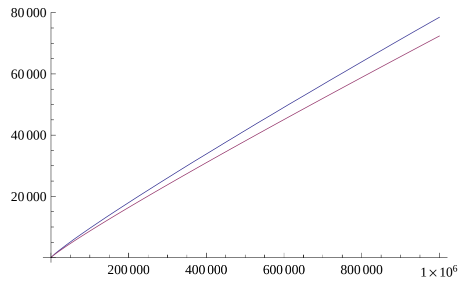
$$\pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$. Here $\pi(x) = \sum_{p \le x} 1$, the number of primes less than or equal to $x$, and $\pi(x) \sim x/\log x$ means that

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1.$$

We can plot $\pi(x)$ against $x/\log x$ :



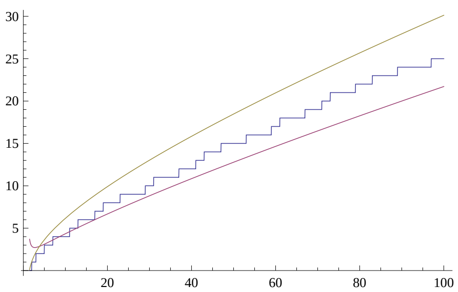



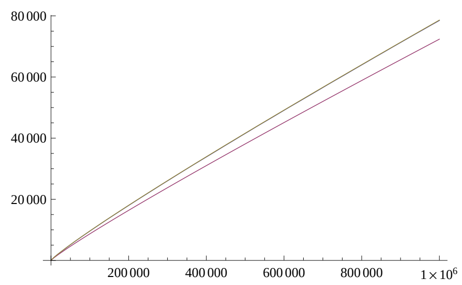

The plot for $\pi(x)$ is in blue, that for $x/\log x$ in red.

Note that $\pi(x)$ is a step function (and thus not continuous) while $x/\log x$ is continuous so there will always be an error between the two. We have discussed in the notes that there is a better approximation to $\pi(x)$ given by the logarithmic integral

$$\mathrm{li}x = \int_2^x \frac{dt}{\log t}.$$

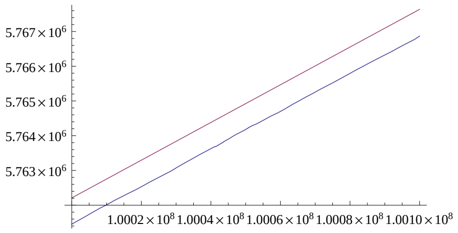
We can plot this function against both $\pi(x)$ and $x/\log x$ :




The plot for li$x$ is in yellow.

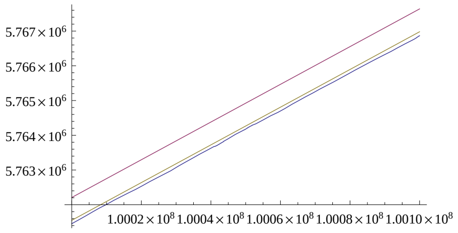
But even li$x$ isn't as good an approximation as can be found. If we look in the range 100,000,000 to 100,100,000 we find a sizeable difference between $\pi(x)$ and li$x$. (li$x$ is now in red).



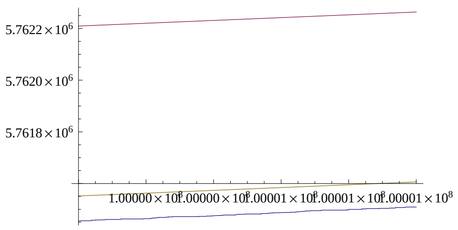

A better approximation was given by Riemann as

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}\left(x^{1/n}\right).$$

If we plot $R(x)$ along with $\pi(x)$ and $\mathrm{li}(x)$ in 100,000,000 to 100,100,000 we find

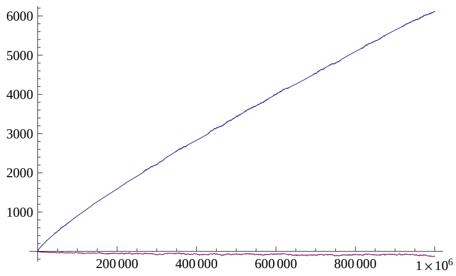



If we zoom in to 100,000,000 to 100,001,000 we find

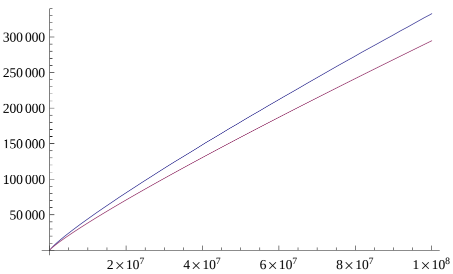



From all these pictures you might guess that $\pi(x) > x/\log x$ and $\pi(x) <$ li$x$ for all $x$. Both of these statements are false, from which you should learn the lesson that you can't tell what will happen as $x \to \infty$ from behaviour for finite $x$ (even if $x$ is as large as 100,100,000!)

Above we have talked of the errors between $\pi(x)$ and $x/\log x$ and $\mathrm{li}x$ so it makes sense to look at the differences $\pi(x) - x/\log x$ and $\pi(x) - \mathrm{li}x$ :




It is $\pi(x) - x/\log x$ that has the largest error. In fact this error is comparable with $x/\log^2 x$ :




It is expected that the error $\pi(x) - \mathrm{li}x$ grows no faster than $Cx^{1/2}\log x$ for some constant $C$.